Journal of Statistical Sciences and Computational Intelligence
e-ISSN: 3092–9458
2(1), 57–74
DOI: https://doi.org/10.64497/jssci.132
https://jssci.org/index.php/pub

Article

Topp-Leone exponential-Lomax distribution: Properties and applications

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Abstract: The Topp-Leone Exponential–Lomax (TLE–L) distribution is a new distribution developed with an improved flexibility when comparing its performance with some other existing and related distributions. This new TLE–L distribution was developed by extending the Lomax distribution with the Topp Leone Exponential G family of distributions. The density and distribution functions of this TLE–L distribution were defined, as well as some respective mathematical properties, such as moments, quantile function, Rényi entropy, and order statistics, which were all derived. The Maximum Likelihood Estimate (MLE) method was considered for the simulation study. The results show that the estimated parameters of this TLE–L distribution are consistent as the BIAS and RMSE approach zero. Also, the model flexibility indicator carried out on TLE – L distribution confirms that it is more flexible compared to its baseline distribution. Conclusively, the TLE–L distribution was applied to model two real data sets; this is done to validate the results obtained from the MLE method only. The results obtained show that the TLE–L distribution best fits the two data sets compared to other distributions used in this study. Possibly, TLE–L distribution would be more useful to fit and model positive real life data sets with the characteristics of high kurtosis and heavy tail towards right hand side (asymmetric data sets).

Keywords: Topp-Leone exponential–Lomax distribution; density and distribution functions; mathematical properties; simulation study; model flexibility indicator; application to real-life data sets.

Received: 10 October 2025; Revised: 12 December 2025; Accepted: 3 January 2026; Published: 28 January 2026

Journal Abbreviation: J. Stat. Sci. Comput. Intell.

1. Introduction

The use of probability distributions in modeling real life time data sets is highly demanding in the field of studies such as applied and physical sciences, engineering, social sciences, medical sciences, agricultural sciences, and many others. This application is done with the concepts of statistical inference, which provide an insight and visualization of the data sets structure and allow making a vital decision on the data distributions.

Thus, among other popularly known probability distribution developed to model asymmetric data sets was Lomax distribution by Lomax [12]. However, this distribution behaves like the Pareto distribution, as it has been applied to fit business failure data in the recent years; this positions it as an alternative to the Pareto distribution because of its importance in modeling economics data sets. Lomax distribution consists of the shape and scale parameter that enables it captures the skewness and kurtosis of any respective data sets.

Researchers such as Balkema and de Haan [6] had extensively used the Lomax distribution for reliability modeling and life testing. Bryson [10]; also used it as an alternative to the exponential distribution when the data are heavy tailed. Ahsanullah [3] studied the record values of the Lomax distribution. Balakrishnan and Ahsanullah [7] introduced some recurrence relations between the moments of record values from the Lomax distribution. Also, the Lomax distribution has been studied, from a Bayesian point of view, by many authors; Nasiri and Hosseini [15], Afaq et al. [4] estimate the parameters of the Lomax distribution using Jeffery's and the extension of Jeffery's prior under different loss functions. More so, Abdullah and Abdullah [2] estimate the parameters of the Lomax distribution based on generalized probability weighted moment, while Al-Awadhi and Ghitany [5] studied the statistical properties of the Poisson Lomax distribution. Although some researchers had extended the Lomax distribution with some families of distributions to form a single compound distribution, such as Abdu-Maniem and Abdel-Hameed [1] who developed the Exponentiated Lomax distribution, Bander and Hannar [8] created the Poisson Lomax distribution, and Ghitany et. al.,[11] developed Mashall-Olkin Extended Lomax distribution, and many others. Although Sanusi et al. [17] studied the exponential distribution with less memory property and hybridized it with the Topp Leone distribution, as the Topp Leone distribution can model a heavy tail data sets. This hybridization is done to have a new family of distributions called the Topp Leone Exponential – G family of distributions. Thereafter, some compound distributions have been developed by extending some existing distributions with the Topp Leone Exponential – G family of distributions due to its nature of robustness and flexibility to adopt and accommodate any baseline distributions. Among the compound distributions created from the said family of distributions by researchers are: Sanusi et. al., [18] studied and developed Topp Leone Exponential – Exponential distribution and Sanusi et. al., [19] developed Topp Leone Exponential – Generalized Inverted Exponential distribution due to its robustness in capturing and improving on the exponential behavior in the data sets So, in this paper, a new asymmetric distribution called Topp Leone Exponential – Lomax (TLE – L) distribution was developed to have a new distribution with the characteristics of Topp Leone, Exponential and Lomax distributions; this was done by extending the Lomax distribution with Top Leone Exponential – G family of distributions.

Thereafter, other sections of this paper were arranged as follows. The methodology used in deriving the cdf, pdf, density graphs, useful expansion, and some properties of the TLE–L distribution were defined and discussed in Section 2. Inference of the TLE–L distribution was derived and discussed in Section 3. Applications to two (2) real life data sets for TLE – L distribution were shown and discussed in Section 4, while Section 5 discusses the conclusion of this paper.

2. Methodology

1.1. Topp Leone Exponential – Lomax Distribution

In this section, we defined the cumulative distribution function (cdf) and probability density function (pdf) of the new distribution called Topp Leone Exponential – Lomax (TLE – L) distribution. However, the cdf and pdf of Topp Leone Exponential G family of distributions developed by sanusi et. al [17] are considered which are both defined as follows:

$$G_{\text{TLE-G}}\left(y;\theta,\phi,\alpha\right)=\left[1-\exp\left\{-2\phi\left(\frac{H\left(y;\alpha\right)}{\bar{H}\left(y;\alpha\right)}\right)\right\}\right]^{\theta} \tag{2.1}$$

and

$$g_{\text{TLE-G}}\left(y;\theta,\phi,\alpha\right)=\frac{\left(2\theta\phi\right)h(y,\alpha)}{\left(\bar{H}\left(y;\alpha\right)\right)^{2}}\exp\left\{-2\phi\left(\frac{H\left(y;\alpha\right)}{\bar{H}\left(y;\alpha\right)}\right)\right\}\left[1-\exp\left\{-2\phi\left(\frac{H\left(y;\alpha\right)}{\bar{H}\left(y;\alpha\right)}\right)\right\}\right]^{\theta-1} \tag{2.2}$$

respectively. Where $h(y,\alpha)$ and $H\left(y;\alpha\right)$ are pdf and cdf of the base line distribution. Also, α is the parameter of the base line distribution. Thus, the cdf and pdf of the Lomax distribution developed by Lomax [12] are given as follows:

$$H\left(y;\pi,\delta\right)=1-\left[1+\left(\frac{y}{\delta}\right)\right]^{-\pi} \tag{2.3}$$

and

$$h\left(y;\pi,\delta\right)=\left(\frac{\pi}{\delta}\right)\left[1+\left(\frac{y}{\delta}\right)\right]^{-\pi-1} \tag{2.4}$$

Respectively, where $y>0,$ also π and δ are both scale and shape parameters.

Note: equation (2.5) below is considered, after the substitution of both equations (2.3) and (2.4) into (2.1) and (2.2).

$$\frac{1-\left[1+\left(\frac{y}{\delta}\right)\right]^{-\pi}}{\left[1+\left(\frac{y}{\delta}\right)\right]^{-\pi}}=\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi}-1 \tag{2.5}$$

Therefore, the cdf of the new distribution called TLE – L distribution is defined as after the input of equation (2.5) in equation (2.1):

$$G_{\text{TLE-L}}(y;\psi)=\left[1-\exp\left\{-2\phi\left(\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi}-1\right)\right\}\right]^{\theta} \tag{2.6}$$

Differentiating the cdf above in equation (2.6), this gives the pdf of Topp Leone Exponential – Lomax (TLE-L) distribution as defined below:

$$g_{TLE-L}(y;\psi)=2\theta\phi\pi\delta^{-1}\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi-1}\exp\left\{-2\phi\left(\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi}-1\right)\right\}\left[1-\exp\left\{-2\phi\left(\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi}-1\right)\right\}\right]^{\theta-1} \tag{2.7}$$

Thus, any random variable Y with the density function defined above in equation (2.7) that follows TLE - $L\left(y,\Psi\right)$ distribution and $\Psi=\left(\theta,\phi,\pi,\delta\right)$ is a vector of parameters; therefore, the survival function

$S\left(y,\Psi\right)$, hazard function $h\left(y,\Psi\right)$, reverse hazard function $\tau\left(y,\Psi\right)$, and cumulative hazard function $H\left(y,\Psi\right)$ for TLE – L distribution are defined as:

$$\mathrm{S}(y;\Psi)=1-\left[1-\exp\left\{-2\phi\left(\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi}-1\right)\right\}\right]^{\theta} \tag{2.8}$$

$$\mathrm{h}\left(y;\Psi\right)=\frac{2\theta\phi\exp\left\{-2\phi\left(\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi}-1\right)\right\}\left[1-\exp\left\{-2\phi\left(\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi}-1\right)\right\}\right]^{\theta-1}}{\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi}\left(1-\left[1-\exp\left\{-2\phi\left(\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi}-1\right)\right\}\right]^{\theta}\right)}. \tag{2.9}$$

$$\tau\left(y;\Psi\right)=\frac{2\theta\phi\exp\left\{-2\phi\left(\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi}-1\right)\right\}}{\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi}\left[1-\exp\left\{-2\phi\left(\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi}-1\right)\right\}\right]}. \tag{2.10}$$

$$\mathrm{H}(y;\Psi)=-\ln\left[1-\left[1-\exp\left\{-2\phi\left(\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi}-1\right)\right\}\right]^{\theta}\right]. \tag{2.11}$$

where $y\varepsilon\Re$

However, the parameters of this new Topp Leone Exponential Lomax distribution are defined as follows: θ is the shape parameter from Topp Leone distribution, ϕ is the scale parameter from the exponential distribution, then π and δ are both scale and shape parameters from the Lomax distribution.

1.2. Shapes of Topp Leone Exponential – Lomax Distribution

The corresponding probability density and hazard functions plots for TLE – L distribution are presented in the figures below. Figure 1 shows the different shapes of data sets, TLE – L distribution can best fit. The data sets are often skewed to the left and L shape data with stretch tail towards the right side. While figure 2 shows the U-shape or bathtub hazard shape of TLE – L distribution.

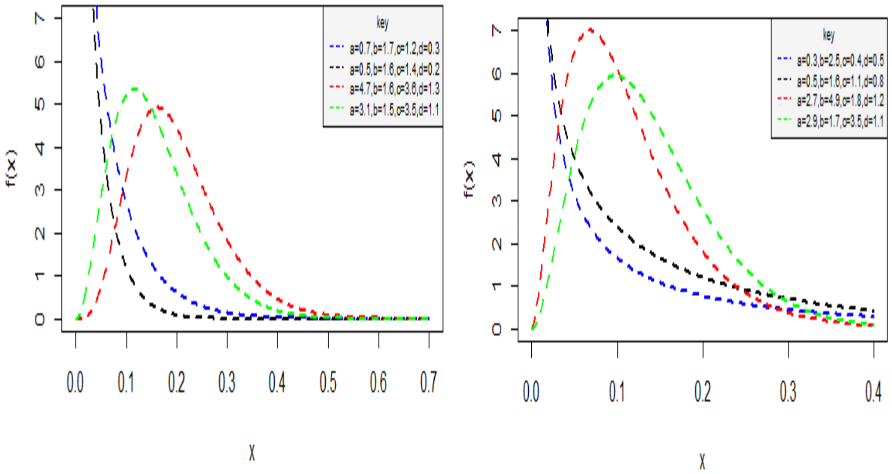

Figure 1. Pdf plots of the TLE – L distribution for different parameter values

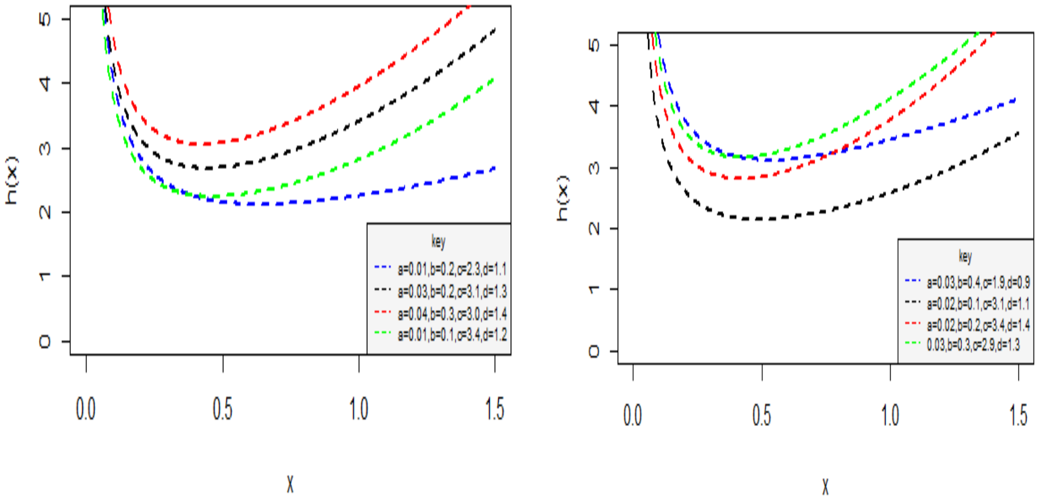

Figure 2. Hazard function plots of the TLE – L distribution for different parameter values.

While the corresponding parameters in the figures above are defined as $\theta = a, \phi = b, \delta = c, \pi = d$.

1.3. Useful Expansion

This section dealt with an important expansion for the pdf of TLE-L distribution by using the idea of generalized binomial and Taylor series expansions in equation (2.12), thus if $|y| \prec 1$ and $a > 0$ is a real non-integer, the power series holds:

$$(1-y)^{k-1} = \sum_{a=0}^{\infty} \frac{(-1)^a \Gamma(k)}{a!\Gamma(k-a)} y^a \tag{2.12}$$

By applying the idea of equation (2.12) on the last term in equation (2.7), the result is;

$$g_{TLE-L}(y)=2\theta\phi\pi\delta^{-1}\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi-1}\left[\exp\left\{-2\phi\left(\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi}-1\right)\right\}\right]^{a+1}\sum_{a=0}^{\infty}(-1)^{a}\frac{\Gamma(\theta)}{a!\Gamma(\theta-a)}$$

$$=2\theta\phi\pi\delta^{-1}\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi-1}\exp\left\{-2\phi(a+1)\left(\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi}-1\right)\right\}\sum_{a=0}^{\infty}(-1)^{a}\frac{\Gamma(\theta)}{a!\Gamma(\theta-a)}$$

$$=2\theta\phi\pi\delta^{-1}\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi-1}\left(\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi}-1\right)^{b}\sum_{a,b=0}^{\infty}(-1)^{a+b}\frac{\Gamma(\theta)\left[2\phi(a+1)\right]^{b}}{a!b!\Gamma(\theta-a)}$$

$$=2\theta\phi\pi\delta^{-1}\sum_{a,b,c=0}^{\infty}(-1)^{a+b+c}\frac{\Gamma(\theta)\left[2\phi(a+1)\right]^{b}b!}{a!b!r!(q-r)!\Gamma(\sigma-p)}\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi\left[1+(b-c)\right]-1}$$

$$=2\theta\phi\pi\delta^{-(1+d)}\sum_{a,b,c,d=0}^{\infty}(-1)^{a+b+c}\frac{\Gamma(\theta)\left[2\phi(a+1)\right]^{b}\Gamma\left(\pi\left[1+(b-c)\right]\right)b!}{a!b!c!d!(b-c)!\Gamma(\theta-a)\Gamma\left(\pi\left[1+(b-c)\right]-d\right)}y^{d}$$

$$=\sum_{a,b,c,d=0}^{\infty}y^{2}\Phi_{a,b,c,d} \tag{2.13}$$

Where $\Phi_{a,b,c,d}=(-1)^{a+b+c}\dfrac{2\theta\phi\pi\delta^{-(1+d)}\Gamma(\theta)\left[2\phi(a+1)\right]^{b}\Gamma\left(\pi\left[1+(b-r)\right]\right)b!}{a!b!c!d!(b-c)!\Gamma(\theta-a)\Gamma\left(\pi\left[1+(b-c)\right]-d\right)}$

1.4. Mathematical Properties

This section provides some mathematical properties of the TLE – L distribution such as moments, quantile function, rényi entropy and order statistics.

1.4.1. Moments

Let Y be a random variable with TLE – L distribution, therefore the raw moment, that is $\mu_n^{'}$, is defined as

$$\mu_n^{'}=E\left(y^n\right)=\int_{-\infty}^{\infty}y^n g_{TLE-L}\left(y;\theta,\phi,\pi,\delta\right)dy \tag{2.14}$$

$g_{TLE-L}\left(y;\theta,\phi,\pi,\delta\right)$ is the pdf of the TLE – L distribution defined in equation (2.7), substitute its expression in equation (2.14) above, and apply the idea of equation (2.12) on its terms that require expansion; thus, we have the expressions below:

$$=\int_{-\infty}^{\infty}y^n 2\theta\phi\pi\delta^{-1}\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi-1}\left[\exp\left\{-2\phi\left(\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi}-1\right)\right\}\right]^{p+1}\sum_{a=0}^{\infty}(-1)^{a}\frac{\Gamma(\theta)}{a!\Gamma(\theta-a)}dy$$

$$= \int_{-\infty}^{\infty} y^n 2\theta\phi\pi\delta^{-1}\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi-1} \exp\left\{-2\phi(a+1)\left(\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi}-1\right)\right\}\sum_{a=0}^{\infty}(-1)^a \frac{\Gamma(\theta)}{a!\Gamma(\theta-a)}dy$$

$$= \int_{-\infty}^{\infty} y^n 2\theta\phi\pi\delta^{-1}\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi-1}\left(\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi}-1\right)^{b} \sum_{a,b=0}^{\infty}(-1)^{a+b}\frac{\Gamma(\theta)\left[2\phi(a+1)\right]^b}{a!b!\Gamma(\theta-a)}dy$$

$$= \int_{-\infty}^{\infty} y^n 2\theta\phi\pi\delta^{-1}\sum_{a,b,c=0}^{\infty}(-1)^{a+b+c}\frac{\Gamma(\theta)\left[2\phi(a+1)\right]^b b!}{a!b!c!(b-c)!\Gamma(\theta-a)}\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi[1+(b-c)]-1} dy$$

$$= \int_{-\infty}^{\infty} y^n 2\theta\phi\pi\delta^{-(1+d)}\sum_{a,b,c,d=0}^{\infty}(-1)^{a+b+c}\frac{\Gamma(\theta)\left[2\phi(a+1)\right]^b\Gamma\left(\pi\left[1+(b-c)\right]\right)b!}{a!b!c!d!(b-c)!\Gamma(\theta-a)\Gamma\left(\pi\left[1+(b-c)\right]-d\right)}y^d dy$$

$$= 2\theta\phi\pi\delta^{-(1+d)}\sum_{a,b,c,d=0}^{\infty}(-1)^{a+b+c}\frac{\Gamma(\theta)\left[2\phi(a+1)\right]^b\Gamma\left(\pi\left[1+(b-c)\right]\right)b!}{a!b!c!d!(b-c)!\Gamma(\theta-a)\Gamma\left(\pi\left[1+(b-c)\right]-d\right)}\int_{-\infty}^{\infty} y^{n+s}dy$$

$$= \sum_{a,b,c,d=0}^{\infty}(-1)^{a+b+c}\frac{2\theta\phi\pi\delta^{-(1+d)}\Gamma(\theta)\left[2\phi(a+1)\right]^b\Gamma\left(\pi\left[1+(b-c)\right]\right)b!}{a!b!c!d!(b-c)!\Gamma(\theta-a)\Gamma\left(\pi\left[1+(b-c)\right]-d\right)}\int_{-\infty}^{\infty} y^{n+s}dy$$

Thus, the expression defined in equation (2.15) below is the moment of the TLE–L distribution:

$$= \sum_{a,b,c,d=0}^{\infty}\Phi_{a,b,c,d}\int_{-\infty}^{\infty} y^{n+d}dy \tag{2.15}$$

Where $\Phi_{a,b,c,d} = (-1)^{a+b+c}\dfrac{2x^s\theta\phi\pi\delta^{-(1+d)}\Gamma(\theta)\left[2\phi(a+1)\right]^b\Gamma\left(\pi\left[1+(b-c)\right]\right)b!}{a!b!c!d!(b-c)!\Gamma(\theta-a)\Gamma\left(\pi\left[1+(b-c)\right]-d\right)}$

1.4.2. Quantile Function

The quantile function of the TLE – G family of distribution is defined as $Q(u) = \mathrm{G}_{\text{TLE-G}}(y;\theta,\phi,\Omega)$, then substitute equation (2.1) in it, and we have the expression below:

$$u = \left[1-\exp\left\{-2\phi\left(\frac{\mathrm{H}(y;\Omega)}{1-\mathrm{H}(y;\Omega)}\right)\right\}\right]^{\theta}$$

$$\exp\left\{-2\phi\left(\frac{\mathrm{H}(y;\alpha)}{1-\mathrm{H}(y;\alpha)}\right)\right\} = 1- u^{\frac{1}{\theta}} \Rightarrow -2\phi\left(\frac{\mathrm{H}(y;\beta)}{1-\mathrm{H}(y;\beta)}\right) = \log\left(1-u^{\frac{1}{\theta}}\right)$$

$$\frac{\mathrm{H}(y;\Omega)}{1-\mathrm{H}(y;\Omega)} = \left(-\frac{1}{2\phi}\right)\log\left(1-u^{\frac{1}{\theta}}\right) \Rightarrow \frac{\mathrm{H}(y;\Omega)}{1-\mathrm{H}(y;\Omega)} = k$$

Where $k = \left(-\frac{1}{2\phi}\right)\log\left(1-u^{\frac{1}{\theta}}\right)$

Thus, $G(x;\Omega) = k\left(1-G(x;\Omega)\right)$

$$\therefore \quad \frac{\mathrm{H}(y;\Omega)}{1-\mathrm{H}(y;\Omega)} = k \quad \Rightarrow \quad \mathrm{H}(y;\Omega) = k\left(1-\mathrm{H}(y;\Omega)\right)$$

$$\mathrm{H}(y;\Omega) = (k-k)\mathrm{H}(y;\Omega) \quad \Rightarrow \quad \mathrm{H}(y;\Omega)+k\left(\mathrm{H}(y;\Omega)\right) = k$$

$$\mathrm{H}(y;\Omega)[1+k] = k$$

$$\text{Hence, } \mathrm{H}(y;\Omega) = \frac{k}{(1+k)} \tag{2.16}$$

Where $\mathrm{H}(y;\Omega)$ is the cdf of the baseline distribution which is the Lomax distribution. Therefore, the quantile function of TLE-G of distributions is derived above in equation (2.16). Then consider equation (2.3) as defined above and substitute it in equation (2.16) in place of $\mathrm{H}(y;\Omega)$, we have:

$$1-\left[1+\left(\frac{y}{\delta}\right)\right]^{-\pi} = \frac{k}{1+k} \quad \Rightarrow \quad \left[1+\left(\frac{y}{\delta}\right)\right]^{-\pi} = 1-\frac{k}{1+k} \quad \Rightarrow \quad \frac{y}{\delta} = \left(1-\frac{k}{1+k}\right)^{-\frac{1}{\pi}}-1$$

Therefore, the quantile function of the TLE–L distribution is defined as:

$$y = \delta\left[\left(1-\frac{k}{1+k}\right)^{-\frac{1}{\pi}}-1\right] \tag{2.17}$$

1.4.3. Rényi Entropy

It is used as an index of diversity and quantifies the uncertainty or randomness of a system. The Rényi entropy for the TLE-L distribution can be defined as

$$I_R(v) = (1-v)^{-1}\log\int_{-\infty}^{\infty} \mathrm{g}_{\text{TLE-L}}^{\mathrm{v}}(y;\theta,\phi,\delta,\pi)\,dy \quad \textit{for } v>0 \textit{ and } v\neq 1$$

where $\mathrm{g}_{\text{TLE-L}}^{\mathrm{v}}(y;\theta,\phi,\delta,\pi)$ is the pdf of TLE – L distribution as defined in equation (2.7) above.

$$\mathrm{g}_{\text{TLE-L}}^{\mathrm{v}}(y) = \left(2\theta\phi\pi\delta^{-1}\right)^{v}\left[1+\left(\frac{y}{\delta}\right)\right]^{v(\pi-1)}\left(\exp\left\{-2\phi\left(\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi}-1\right)\right\}\right)^{v}\left[1-\exp\left\{-2\phi\left(\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi}-1\right)\right\}\right]^{v(\theta-1)}$$

https://doi.org/10.64497/jssci.132

$$=\left(2\theta\phi\pi\delta^{-1}\right)^{v}\left[1+\left(\frac{y}{\delta}\right)\right]^{v(\pi-1)}\left(\exp\left\{-2\phi\left(\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi}-1\right)\right\}\right)^{v+a}\sum_{a=0}^{\infty}(-1)^{a}\frac{\left[v(\theta-1)\right]!}{a!\left[v(\theta-1)-a\right]!}$$

$$=\left(2\theta\phi\pi\delta^{-1}\right)^{v}\left[1+\left(\frac{y}{\delta}\right)\right]^{v(\pi-1)}\left(\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi}-1\right)^{b}\sum_{a,b=0}^{\infty}(-1)^{a+b}\frac{\left[v(\theta-1)\right]!\left[v+a\right]^{b}(2\phi)^{b}}{a!b!\left[v(\theta-1)-a\right]!}$$

$$=\left(2\theta\phi\pi\delta^{-1}\right)^{v}\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi(v+b-c)-v}\sum_{a,b,c=0}^{\infty}(-1)^{a+b+c}\frac{\left[v(\theta-1)\right]!\left[v+a\right]^{b}(2\phi)^{b}b!}{a!b!c!\left[v(\sigma-1)-a\right]!\left[b-c\right]!}$$

$$=\sum_{a,b,c,d=0}^{\infty}(-1)^{a+b+c}\frac{(2\theta\phi\pi)^{v}\delta^{-(v+s)}\left[v(\theta-1)\right]!\left[v+a\right]^{b}(2\phi)^{b}b!\left[\pi(v+b-c)-v\right]!}{a!b!c!d!\left[v(\theta-1)-a\right]!\left[b-c\right]!\left[\left(\pi(v+b-c)-v\right)-d\right]!}y^{d}$$

Thus, $\mathrm{g}_{\text{TLE-L}}^{\mathrm{v}}(y)=\sum_{a,b,c,d=0}^{\infty}y^{2}\Pi_{a,b,c,d}$,

$$\text{Therefore, } I_{R}(v)=(1-v)^{-1}\log\sum_{a,b,c,d=0}^{\infty}\Pi_{a,b,c,d}\int_{-\infty}^{\infty}y^{s}dy \tag{2.18}$$

$$\text{where }\quad \Pi_{a,b,c,d}=(-1)^{a+b+c}\frac{(2\theta\phi\pi)^{v}\delta^{-(v+d)}\left[v(\theta-1)\right]!\left[v+a\right]^{b}(2\phi)^{b}b!\left[\pi(v+b-c)-v\right]!}{a!b!c!d!\left[v(\theta-1)-a\right]!\left[b-c\right]!\left[\left(\pi(v+b-c)-v\right)-d\right]!}$$

1.4.4. Order Statistics

Given a random vector $\mathrm{Y}_1\ \mathrm{Y}_2\ .\ .\ .\ \mathrm{Y}_\mathrm{n}$ on the probability space $(\mathrm{S}\ \in\ \mathrm{P})$, for each $\mathrm{s}\in\mathrm{S}$, sort it in the component vector $\left[\mathrm{Y}_{(1)}(\mathrm{S}),\mathrm{Y}_{(2)}(\mathrm{S})\ .\ .\ .\mathrm{Y}_{(\mathrm{n})}(\mathrm{S})\right]$ satisfying $\mathrm{Y}_{(1)}(\mathrm{S})\leq\mathrm{Y}_{(2)}(\mathrm{S})\ \leq.\ .\ .\leq\mathrm{Y}_{(\mathrm{n})}(\mathrm{S})$. Therefore, the vector $\left(\mathrm{Y}_{(1)},\mathrm{Y}_{(2)}.\ .\ .\mathrm{Y}_{(\mathrm{n})}\right)$ is called vector of order statistics of $\mathrm{Y}_{(1)},\mathrm{Y}_{(2)}.\ .\ .\mathrm{Y}_{(\mathrm{n})}$. Thus, the order statistics is defined as follows

$\mathrm{f}_{\mathrm{i,n}}=\dfrac{g_{\text{TLE-L}}(y;\psi)G_{\text{TLE-L}}^{j+i-1}(y;\psi)}{\beta(i,n-i+1)}\sum_{j=0}^{n-i}(-1)^{j}\binom{n-1}{j}$ where $g_{\text{TLE-L}}(y;\psi)$ and $G_{\text{TLE-L}}^{j+i-1}(y;\psi)$ are both the pdf and cdf of the TLE – L distribution. Also, $\psi=\theta,\phi,\delta,\pi$ Therefore, apply the idea of equation (2.12) on their terms that require expansion; thus, we have the expressions below:

$$\mathrm{g}_{TLE-L}(y;\psi)G_{TLE-L}{}^{j+i-1}(y;\psi)=2\theta\phi\pi\delta^{-1}\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi-1}\exp\left\{-2\phi\left(\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi}-1\right)\right\}$$

$$\times\left[1-\exp\left\{-2\phi\left(\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi}-1\right)\right\}\right]^{\theta\left[1+(j+i-1)\right]-1}$$

$$= 2\theta\phi\pi\delta^{-1}\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi-1}\exp\left\{-2\phi\left(a+1\right)\left(\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi}-1\right)\right\}\sum_{a=0}^{\infty}\left(-1\right)^{a}\frac{\Gamma\left(\theta\left[1+\left(j+i-1\right)\right]\right)}{a!\Gamma\left(\theta\left[1+\left(j+i-1\right)\right]-a\right)}$$

$$= 2\theta\phi\pi\delta^{-1}\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi-1}\left(\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi}-1\right)^{b}\sum_{a,b=0}^{\infty}\left(-1\right)^{a+b}\frac{\left(2\phi\right)^{b}\left(a+1\right)^{b}\Gamma\left(\theta\left[1+\left(j+i-1\right)\right]\right)}{a!b!\Gamma\left(\theta\left[1+\left(j+i-1\right)\right]-p\right)}$$

$$= 2\theta\phi\pi\delta^{-1}\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi\left(1+b\right)-1}\sum_{a,b,c=0}^{\infty}\left(-1\right)^{a+b+c}\frac{\left(2\phi\right)^{b}\left(a+1\right)^{b}\Gamma\left(\theta\left[1+\left(j+i-1\right)\right]\right)b!}{a!b!c!\Gamma\left(\theta\left[1+\left(j+i-1\right)\right]-a\right)\left(b-c\right)!}$$

Then, $g_{TLE-L}(y;\psi)G_{TLE-L}{}^{j+i-1}\left(y;\psi\right)=\sum_{a,b,c,d=0}^{\infty}\left(-1\right)^{a+b+c}\frac{\left(2\phi\right)^{b+1}\theta\pi\delta^{-\left(1+s\right)}\left(a+1\right)^{b}\Gamma\left(\theta\left[1+\left(j+i-1\right)\right]\right)b!}{a!b!c!d!\Gamma\left(\theta\left[1+\left(j+i-1\right)\right]-a\right)\left(b-c\right)!}$

$$\times\frac{\Gamma\left[\pi\left(1+b\right)\right]}{\Gamma\left[\pi\left(1+b\right)-d\right]}y^{s}$$

So, the order statistics of TLE – L distribution is defined below in equation (2.19)

$$\text{Therefore, } f_{i,n}=\frac{\binom{n-1}{j}}{\beta\left(i,n-i+1\right)}\sum_{j=0}^{n-i}\left(-1\right)^{j}\sum_{p,q,r,s=0}^{\infty}y^{s}\Upsilon_{a,b,c,d}$$

$$\text{Where } \Upsilon_{a,b,c,d}=\left(-1\right)^{a+b+c}\frac{\left(2\phi\right)^{b+1}\theta\pi\delta^{-\left(1+s\right)}\left(a+1\right)^{b}\Gamma\left(\theta\left[1+\left(j+i-1\right)\right]\right)b!\Gamma\left[\pi\left(1+b\right)\right]}{a!b!c!d!\Gamma\left(\theta\left[1+\left(j+i-1\right)\right]-a\right)\left(b-c\right)!\Gamma\left[\pi\left(1+b\right)-d\right]} \tag{2.19}$$

3. Inference

Here, an inferential study of the TLE – L distribution is proposed, as well simulation study and a model flexibility indicator.

3.1. Parameter Estimation

Many approaches are used for parameter(s) estimation in some distributions; however, the maximum likelihood method remains the most commonly used estimation method among others. Thus, the maximum likelihood estimators of the unknown parameters of the TLE – L distribution from complete samples are determined. Let $Y_1,\ldots,Y_n$ be observed values from the TLE – L distribution with a vector of parameters Ψ. The Log-likelihood function of the pdf defined in equation (2.7) can be expressed as:

$$l(\Psi)=n\log\left(2\right)+n\log\left(\theta\right)+n\log\left(\phi\right)+n\log\left(\pi\right)-n\log\left(\delta\right)+\left(\pi-1\right)\sum_{i=1}^{n}\log\left[1+\left(\frac{y}{\delta}\right)\right]-2\phi\sum_{i=1}^{n}\left(\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi}-1\right)$$
$$+\left(\theta-1\right)\sum_{i=1}^{n}\log\left[1-\exp\left\{-2\phi\left(\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi}-1\right)\right\}\right] \tag{3.1}$$

Equation (3.1) above is differentiated with respect to each of these parameters $\theta, \phi, \delta, \pi$ and equivalently equate each of them to 0, thus we have as follows:

$$\frac{n}{\theta}+\sum_{i=1}^{n}\log\left[1-\exp\left\{-2\phi\left(\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi}-1\right)\right\}\right] = 0$$

$$\frac{n}{\phi}-2\sum_{i=1}^{n}\left(\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi}-1\right)-2(\theta-1)\sum_{i=1}^{n}\left(\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi}-1\right)\frac{\exp\left\{-2\phi\left(\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi}-1\right)\right\}}{\left[1-\exp\left\{-2\phi\left(\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi}-1\right)\right\}\right]} = 0$$

$$\frac{n}{\pi}+\sum_{i=1}^{n}\log\left[1+\left(\frac{y}{\delta}\right)\right]-2\phi\left(\frac{y}{\delta}\right)\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi}\sum_{i=1}^{n}\left\{\left(\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi}-1\right)\right\}+(\theta-1)\left(\frac{y}{\delta}\right)\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi}\sum_{i=1}^{n}\frac{\exp\left\{-2\phi\left(\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi}-1\right)\right\}}{\left[1-\exp\left\{-2\phi\left(\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi}-1\right)\right\}\right]} = 0$$

$$-\frac{n}{\delta}+(\pi-1)y\sum_{i=1}^{n}\frac{\left(\frac{y}{\delta}\right)}{\left[1+\left(\frac{y}{\delta}\right)\right]}-2\phi y\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi}\sum_{i=1}^{n}\left(\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi}-1\right)+y(\theta-1)\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi}\sum_{i=1}^{n}\frac{\exp\left\{-2\phi\left(\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi}-1\right)\right\}}{\left[1-\exp\left\{-2\phi\left(\left[1+\left(\frac{y}{\delta}\right)\right]^{\pi}-1\right)\right\}\right]} = 0$$

3.2. Simulation Study

The study of Simulation is carried out for different sample size on the efficiency of MLE method. This is done to determine the parameters' consistency of Topp Leone Exponential – Lomax (TLE – L) distribution. R software is used to generate a random variable X from Topp Leone Exponential – Lomax (TLE – L) distribution. Thus, the size n = 10; 20; 40; 60; 80 and 100 from Topp Leone Exponential – Lomax (TLE – L) distribution was sampled for some selected combination of parameters were used. The process is done for N = 1000 time to calculate mean estimate, Bias and Root Mean Square Error (RMSE). The outcome obtained is given in table 1 below. The respective initial values used for the four parameters of Topp Leone Exponential – Lomax (TLE – L) distribution are $\theta=5.5, \phi=1.9, \delta=1.1, \pi=1.1$ and $\theta=5.5, \phi=1.9, \delta=1.0, \pi=1.0$. Thus, these values are chosen based on the suitable values that best fit the initial values to initiate the parameters behavior. It is observed, that, the higher the sample size the closer the mean values are to the initial values. Also, the values of the Bias and RMSE (Root Mean Square Error) approach zero. The results obtained from the simulation study on this new distribution called TLE – L distribution are presented in the table 1 below. Therefore, TLE – L distribution is consistent. Note, θ = Shape parameter, ϕ = Scale parameter, δ = Scale parameter and π = Shape parameter.

Table 1. Means, Bias and RMSEs for the respective TLE – L when $\theta = 5.5$ and $\phi = 1.9$.

		Set I	**δ=1.1**	**π=1.1**		**Set II**	**δ=1.0**	**π=1.0**	
n		**Parameter**				**Parameter**			
		θ	φ	δ	π	θ	φ	δ	π
10	Mean	8.0689	0.7947	1.1691	2.1351	7.7583	0.7687	1.1288	2.0278
	Bias	2.5689	-1.1053	0.0691	1.0351	2.2583	-1.1313	0.1288	1.0278
	RMSE	3.6744	1.2775	0.7102	1.1548	3.2397	1.2881	0.8577	1.1583
20	Mean	7.6860	0.7359	1.1100	2.0936	7.4402	0.6887	1.0348	1.9838
	Bias	2.1860	-1.1641	0.0100	0.9936	1.9402	-1.2113	0.0348	0.9838
	RMSE	3.0239	1.2781	0.6049	1.1022	2.6416	1.2998	0.7090	1.0902
40	Mean	7.4362	0.6611	1.0240	2.0445	7.2295	0.6227	0.9362	1.9254
	Bias	1.9362	-1.2389	-0.0760	0.9445	1.7295	-1.2773	-0.0638	0.9254
	RMSE	2.4410	1.3009	0.4890	1.0225	2.2075	1.3241	0.4473	0.9689
60	Mean	7.3988	0.6276	0.9828	2.0297	7.1441	0.5830	0.8842	1.9143
	Bias	1.8988	-1.2724	-0.1172	0.9297	1.6441	-1.3170	-0.1158	0.9143
	RMSE	2.3536	1.3213	0.4477	0.9894	1.9423	1.3456	0.2771	0.9326
100	Mean	7.3098	0.5971	0.9642	2.0180	7.1987	0.5491	1.9135	0.8597
	Bias	1.8098	-1.3029	-0.1358	0.9180	1.6987	-1.3509	0.9135	-0.1403
	RMSE	2.2068	1.3290	0.4061	0.9621	2.0023	1.3649	0.9259	0.2388

3.3. Model Flexibility Indicators

The flexibility of Topp Leone Exponential – Lomax (TLE – L) distribution with respect to its baseline (Lomax) distribution with different values of their parameters was discussed. The skewness and kurtosis of the new distribution and its respective baseline distributions are computed using: Bowley [9] skewness and Moors [14] kurtosis, which are defined as

$$S = \frac{Q\left(\frac{3}{4}\right) - 2Q\left(\frac{1}{2}\right) + Q\left(\frac{1}{4}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)} \tag{3.2}$$

$$K = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) - Q\left(\frac{3}{8}\right) + Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)} \tag{3.3}$$

respectively.

The table 2 below reports the skewness and kurtosis of Topp Leone Exponential (TLE – L) distribution and its respective baseline distribution in order to assess its flexibility compare to the base line distribution.

It is also reported in table 2 that, for some corresponding parameter values. The skewness of TLE – L distribution ranges from 10.5410 to 13.9652 and kurtosis is between 12.0048 and 16.4830 and the skewness of the baseline distribution ranges from 2.9912 to 5.2541 and the Kurtosis is between 3.3619 and 5.9757. In terms of skewness and kurtosis, it is clear that TLE – L distribution is more flexible than the baseline distribution that is, Lomax distribution.

Table 2. Skewness and Kurtosis of TLE-L and L distributions using different parameter values

Parameters				**TLE – L**		**L**	
				Skewness	**Kurtosis**	**Skewness**	**Kurtosis**
θ	φ	δ	π	10.8507	12.5000	5.2541	5.9757
0.1	0.4	0.5	0.1	15.5540	17.6891	4.8892	5.4696
0.1	0.4	0.5	0.2	10.5410	12.0048	2.9912	3.3619
0.1	0.2	5	2	13.9652	16.4830	2.9912	3.3619
0.1	0.4	5	2	12.0743	13.8853	2.9717	3.2839
0.1	0.4	5	3	10.8507	12.5000	5.2541	5.9757

4. Application

4.1. Application to Data Sets

Application to two real life data sets to confirm the flexibility of Topp Leone Exponential – Lomax (TLE – L) distribution is done by comparing its performance with some other existing distributions. Maximum likelihood estimation method is used to estimate the parameters of TLE – L distribution. Thus, Tables 3 and 4 present the MLE's for models' parameter and goodness-of-fit statistic showing the distribution that best fit the two data sets. In comparing the fitted models, the goodness-of-fit measures popularly known that is; Akaike information criterion (AIC) and Bayesian information criterion (BIC) were considered. The distribution among others with the lowest value of AIC & BIC that best fit the data sets is shown in the two tables 3 & 4 with respective discussion below each table.

4.2. The Comparators

These are the pdf of the existing distributions compared with TLE – L distribution in fitting the two data sets.

Topp Leone Exponentiated – Lomax (TLET – L) distribution:

$$\mathrm{f}\left(y\right)=2\sigma\varpi\left(\frac{\varphi}{\alpha}\right)\left[1+\frac{y}{\alpha}\right]^{-(\varphi+1)}\left[1-\left[1+\frac{y}{\alpha}\right]^{-\varphi}\right]^{\sigma-1}\left[1-\left[1-\left[1+\frac{y}{\alpha}\right]^{-\varphi}\right]^{\sigma}\right]\left[1-\left(1-\left[1-\left[1+\frac{y}{\alpha}\right]^{-\varphi}\right]^{\sigma}\right)\right]^{\varpi-1}$$

Topp Leone Odd Lindley – Lomax (TLOL – L) distribution:

$$\mathrm{f}\left(y\right)=\frac{2\sigma\psi^{2}\alpha\varphi}{\left(1+\psi\right)^{2}}\left(1+\varphi y\right)^{3\alpha-1}\left[\psi+\left(1+\varphi y\right)^{-\alpha}\right]\exp\left\{-2\psi\left(\left(1+\varphi y\right)^{\alpha}-1\right)\right\}$$

$$\times\left[1-\left(\frac{\psi+\left(1+\varphi y\right)^{-\alpha}}{\left(1+\psi\right)\left(1+\varphi y\right)^{-\alpha}}\right)^{2}\exp\left\{-2\psi\left(\left(1+\varphi y\right)^{\alpha}-1\right)\right\}\right]^{\sigma-1}$$

Topp Leone Exponentiated - Inverse exponential (TLET – IE): distribution:

$$\mathrm{f}(y)=2\varpi\sigma\left(\frac{\pi}{y^2}\right)\left(\exp\left\{-\left(\frac{\pi}{y}\right)\right\}\right)^{\varpi}\left[1-\left(\exp\left\{-\left(\frac{\pi}{y}\right)\right\}\right)^{\varpi}\right]\left[1-\left[1-\left(\exp\left\{-\left(\frac{\pi}{y}\right)\right\}\right)^{\varpi}\right]^2\right]^{\sigma-1}$$

Topp Leone-Inverse Exponential (TL – IE) distribution:

$$\mathrm{f}(y)=2\sigma\left(\frac{\pi}{y^2}\right)\exp\left\{-\left(\frac{\pi}{y}\right)\right\}\left\{1-\exp\left\{-\left(\frac{\pi}{y}\right)\right\}\right\}\left[1-\left(1-\exp\left\{-\left(\frac{\pi}{y}\right)\right\}\right)^2\right]^{\sigma-1}$$

Exponential Lomax (EL) distribution:

$$\mathrm{f}(y)=\frac{\lambda\varphi\left[1+\dfrac{y}{\alpha}\right]^{-(3\varphi+1)}}{\alpha}\exp\left\{-\lambda\left(\left[1+\left(\frac{y}{\alpha}\right)\right]^{\varphi}-1\right)\right\}$$

Lomax (L) distribution:

$$\mathrm{f}(y)=\frac{\varphi}{\alpha}\left(1+\frac{y}{\alpha}\right)^{-(\varphi+1)}$$

Data set 1

Data set 1 represents the survival times of 121 patients with breast cancer. This is gotten from a large hospital between a period of 1929 & 1938 which has been previously used by Lee [13]and Ramos et al., [16].

Data set 2

Data set represents the death times in weeks of patients with cancer of the tongue with that is aneuploidy DNA profile. This data set has been used as well by Sickle-Santanello et. al., [20].

Table 3 MLEs with the Goodness-of-fit test criteria on the data set 1 of patients with breast cancer

Models	Estimates	-LL	AIC	BIC
TLE – L	θ=1.9019 ϕ = 0.0311 π = 0.8843 δ =1.2882	581.4892	1170.9780	1171.3095
TLET – IE	σ =7.6935 λ =1.2472 γ =1.4595	751.1839	1508.3680	1508.6161
TL – IE	σ =0.9288 π =11.2290	753.5886	1511.1770	1511.3428
EL	λ =8.2951e+08 φ =0.0015 α =5.6577e+07	585.1278	1176.2560	1176.5039
L	φ =4.5201e+04 α =-2.0944e+04	585.1283	1174.2570	1174.4222

In Table 3 above, the distribution with the lowest AIC & BIC is TLE – L distribution (1170.9780 & 1171.3095) followed by TLET – IE distribution (1508.3680 & 1508.6161) followed by TL – IE distribution (1511.1770 & 1511.3428) followed by EL distribution (1176.2560 & 1176.5039) and lastly L distribution (1174.2570 & 1174.4222), this shows that TLE – L distribution best fit the data sets.

Table1 4. MLEs with the Goodness-of-fit test criteria on the data set 2 of patients with cancer of the tongue

Models	Estimates	-LL	AIC	BIC
TLE – L	θ=0.9926 ϕ=0.0960 π =1.3421 δ =30.1853	273.8697	553.7394	554.569
TLET-L	σ =0.0823 ϖ =45.5710 φ=5.8667 α =0.0007	277.3826	558.7653	561.5955

Also, in Table 4, TLE – L distribution with the AIC and BIC values of 553.7394 & 554.5697 respectively best fit the data set, followed by TLET – L distribution (558.7653 & 561.5955) followed by EL distribution (689.6753 & 688.7981), then TOL – L distribution (960.9554 & 959.7857). A significant proved that TLE – L distribution best fit the data set.

The Figure 3 below shows how TLE – L distribution best fit the two data sets, the deep black line on the two graph is the line of TLE – L distribution indicating how the distribution perfectly capture the two data sets.

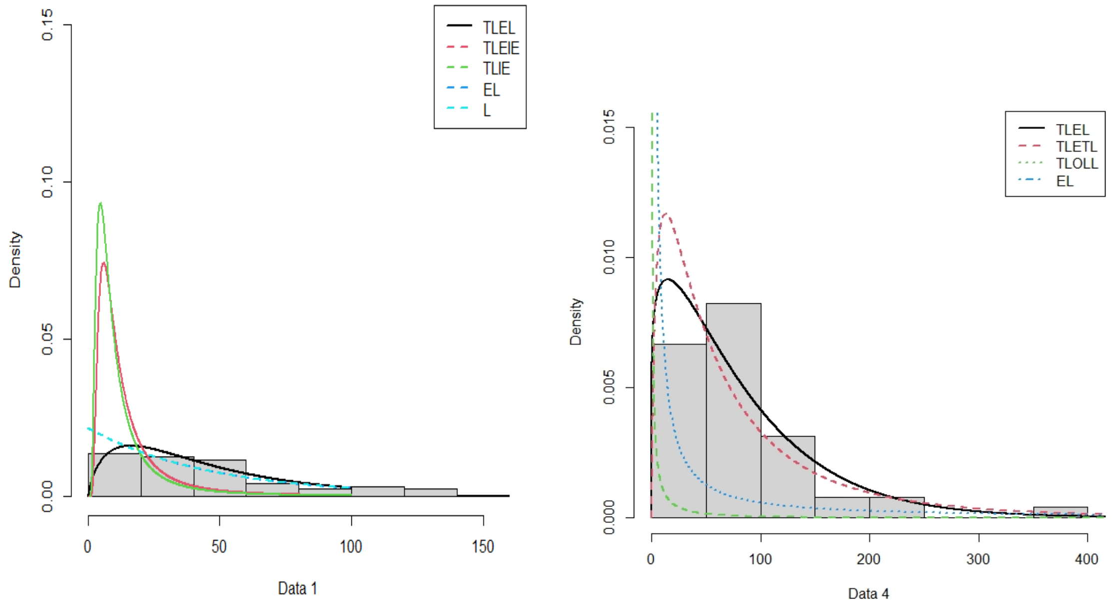

Figure 3. Fitted pdfs for the TLE – L, TLE – IE, TL – IE, TLET – L, TLOL – L, EL, and L models to the data sets 1 and 2.

5. Conclusion

A new distribution called the Topp-Leone Exponential – Lomax (TLE – L) distribution was developed. This is done by extending the Lomax distribution with the Topp–Leone Exponential G family of distributions. This new distribution possesses the characteristics of Topp Leone, exponential, and Lomax distributions with four distinctive parameters. Furthermore, the respective mathematical properties of this compound distribution were derived with the idea of the binomial theorem expansion. A simulation study with the method of maximum likelihood estimation proved that the four parameters are consistent. The flexibility model test showed that this new TLE – L distribution is more flexible compared to its baseline distribution and some respective competitive distributions. Conclusively, this new TLE – L distribution was applied to two real data sets to assess its robustness in flexibility over the existing distributions. It was significantly observed that this new distribution best fit the two data sets compared to the other distributions used, among which are Topp Leone Exponentiated Lomax, which possesses only the characteristics of Topp Leone exponentiated and lomax; and Topp Leone Odd Lindley – Lomax distribution which possesses only the characteristics of Topp Leone odd Lindley and Lomax. Thus, it was noted that the additional characteristics of exponential distribution in this new distribution called Topp Leone Exponential – Lomax (TLE – L) distribution helps it to capture the exponential trend in the two data sets which other distributions perhaps cannot capture; with this additional characteristic feature, this has increase and boosts the flexibility of this new distribution called Topp Leone Exponential - Lomax (TLE – L) distribution. Thus, this new TLE – L distribution is limited to model data with high kurtosis with heavy tail distribution, but may not be able to model data sets with the characteristics of trigonometric trends. Therefore, there is need to extend this new distribution with any of the trigonometric family of distributions to confirm its ability of flexibility in modeling trigonometric data sets.

Author contribution: A. A Sanusi derived the methodology, inference and done the applications part of the work. While, S. I Doguwa cross checked and designed the arrangement of the respective equations and A. Yahaya confirmed the write up on the introduction and did the comprehensive checking on the entire research work.

Funding Statement: No any form of funding attached to this research work; however it is done under personal funding and efforts.

Data Availability:

Data Set 1

The survival times of 121 patients with breast cancer obtained from a large hospital in a period from 1929 – 1938, below is the data set by Lee [13]and Ramos *et al.,* [16].

0.3, 0.3, 4.0, 5.0, 5.6, 6.2, 6.3, 6.6, 6.8, 7.4, 7.5, 8.4, 8.4, 10.3, 11.0, 11.8, 12.2, 12.3, 13.5, 14.4, 14.4, 14.8, 15.5, 15.7, 16.2, 16.3, 16.5, 16.8, 17.2, 17.3, 17.5, 17.9, 19.8, 20.4, 20.9, 21.0, 21.0, 21.1, 23.0, 23.4, 23.6, 24.0, 24.0, 27.9, 28.2, 29.1, 30.0, 31.0, 31.0, 32.0, 35.0, 35.0, 37.0, 37.0, 37.0, 38.0, 38.0, 38.0, 39.0, 39.0, 40.0, 40.0, 40.0, 41.0, 41.0, 41.0, 42.0, 43.0, 43.0, 43.0, 44.0, 45.0, 45.0, 46.0, 46.0, 47.0, 48.0, 49.0, 51.0, 51.0, 51.0, 52.0, 54.0, 55.0, 56.0, 57.0, 58.0, 59.0, 60.0, 60.0, 60.0, 61.0, 62.0, 65.0, 65.0, 67.0, 67.0, 68.0, 69.0, 78.0, 80.0, 83.0, 88.0, 89.0, 90.0, 93.0, 96.0, 103.0, 105.0, 109.0, 109.0, 111.0, 115.0, 117.0, 125.0, 126.0, 127.0, 129.0, 129.0, 139.0, 154.0.

Data set 2:

It represents the death times (in weeks) of patients with cancer of the tongue with aneuploidy DNA profile, below is the data set by Sickle-Santanello *et. al.*, [20].

1, 3, 3, 4, 10, 13, 13, 16, 16, 24, 26, 27, 28, 30, 30, 32, 41, 51, 61, 65, 67, 70, 72, 73, 74, 77, 79, 80, 81, 87, 87, 88, 89, 91, 93, 96, 97, 100, 101, 104, 104, 108, 109, 120, 131, 150, 157, 167, 231, 240, 400.

Acknowledgements: I recommend and appreciate the efforts of my mentors and the supports of my parent, family, relatives, friends and colleagues in various dimension that significantly contribute to the success of this research work may Almighty Allah continue to bestow His mercy and blessings upon every one of us ameen.

Conflict of Interest: This research work has not been sent to any journal for neither review nor publication, it is free from any copy right, it is a work done extracted from the corresponding Author's PhD thesis written on two sets of families of distributions.

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